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On the similarity solution of the fragmentation equation

Richard P Treat

Physics Department, West Virginia University, Morgantown, West Virginia 26506, USA

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Abstract. An analysis of the similarity solution of the linear, homogeneous, fragmentation equation is given for a general volume-conserving daughter-fragment distribution. For the special case of a polynomial daughter-distribution of degree p, an exact, basic similarity solution for the time evolution of the particle-volume distribution is derived. The solution is proportional to the $G^{p+1,0}_{p,p+1}$ Meijer G-function which may be represented as a linear combination of ${}_pF_p$ generalized hypergeometric functions. The properties of generalized hypergeometric functions and G-functions are given in the special function literature and include the continuation of the solution to large values of the similarity variable which is given here for special cases.

1. Introduction

In the continuous, linear theory of the fragmentation (or splitting) of particles the actual discreteness of the particles is ignored and it is assumed that the particles fragment by some perturbation other than a collision with another particle of the same kind. Furthermore, it is assumed that there is no recombination of the particles, that the rate of fragmentation of a particle of volume x is proportional to the concentration of particles of volume x and that changes in the particle-volume distribution are brought about only by fragmentation. Other effects that could change the distribution, such as spatial diffusion caused by concentration gradients or convective transport, are assumed to be negligible. These assumptions lead to a linear differential–integral equation called the fragmentation equation that governs the evolution in time of the distribution of the particle volumes in a system of fragmenting of particles.

Here we study the solution of the fragmentation equation for a homogeneous fragmentation kernel and boundary conditions on the particle volumes that assure that the total volume of particles remains constant as the fragmentation proceeds. The kinetics of fragmentation with volume change is also of interest and has been studied by Fillipov [1], McGrady and Ziff [2], Edwards *et al* [3], Cai *et al* [4], Huang *et al* [5], Ernst and Szamel [6] and Said and El-Wakil [7], but in this investigation volume is conserved. The studies that are most relevant to the analysis presented here are those by Fillipov [1], Ziff and McGrady [8, 9], Peterson [10], Cheng and Redner [11, 12], Huang *et al* [5], Ziff [13] and Baumann *et al* [14]. Additional references, especially to earlier work, are given in papers by Ziff [13] and Bak and Bak [15]. Possible applications of the theory to polymer degradation, droplet and aggregate breakage, combustion and other physical processes have been pointed out in [5, 8–12]. A correspondence of the linear fragmentation theory to a one-dimensional Markov process has also been shown by Fillipov [1].

The studies referred to above have been on large and small particle-volume limits, on the long-time limits of the particle distribution and the moments of the particle distribution

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and on the construction of exact solutions of the fragmentation equation. This investigation continues on these same problems in the context of the similarity form of solution for a fragmentation probability that is homogeneous in the particle volume. In the theory of coagulation [16, 17], and also in the theory of fragmentation, a similarity solution has the same invariances as the governing equations. An advantage in finding the form of the similarity solution is that it separates the independent variables and reduces the problem to the solution of separated ordinary differential equations. The equations that govern the kinetics of particle coagulation and fragmentation are invariant under time translations and, if the kernels that give the probability of coagulation and fragmentation are homogeneous functions of the particle volume, these equations are also invariant under a group of scale transformations of the particle volume and time. Friedlander and Wang [18], guided by experimental results and intuition, have given an ansatz for the form of the similarity solution for the coagulation equation and, even though the coagulation equation is nonlinear and the fragmentation equation is linear, since the invariances of the coagulation and fragmentation equations are the same, the Friedlander ansatz also brings about the separation of variables in the fragmentation theory.

Although the general form of the similarity solutions for coagulation and fragmentation is exactly the same, the solutions are of course different. For coagulation the similarity solution is a solution of a nonlinear equation that is distinct from other solutions that are not scale invariant. Special importance is attached to the similarity solution because, in addition to the separation of variables that it brings about, in the case of the constant coagulation kernel [17–19] (and probably for other kernels too [17–21]), it is the limit to which solutions with different initial values approach after a sufficiently long time.

The situation is partly the same for fragmentation where, as shown by Fillipov [1] and Ziff and McGrady [8], solutions with different initial values tend to the limit of a similarity solution. However, an essential difference in the two theories is that for fragmentation, because of the linearity of the theory, one can add solutions to obtain more general solutions. In particular, one can add basic similarity solutions or add solutions for the monodisperse initial distribution where the latter method is a way to form the general solution of the initial value problem.

We commented above on coagulation in order to bring in the origin of the Friedlander ansatz, but the analysis here is only on fragmentation. First, rather than use an ansatz, the Friedlander form of the similarity solution is derived from an invariance argument. The similarity form is derived by constructing a solution that has the same time translation and scale invariance as the equation itself. Then, for a general, homogeneous kernel, we study the basic similarity solution and the moments of the solution. The large and small mean-particle-volume limits that have been given by Cheng and Redner [11, 12] are derived here as the limits of the basic solution. Furthermore, continuing the studies of Fillipov [1], Ziff and McGrady [8, 9], Peterson [10], Ziff [13], Baumann *et al* [14] and Said and El-Wakil [7] we consider the problem of finding exact similarity solutions of the fragmentation equation that hold initially and for large times as well. Assuming a polynomial class of fragmentation kernel, a basic similarity solution is derived by the Mellin transformation that is a generalization of the solutions that have previously been given. The solution shows the existence of solutions in a way that is different than the proof given by Fillipov [1] and shows explicitly conditions on the fragmentation parameters that are sufficient for the existence of the solution. The solution is in agreement with the general results on small and large limits of the similarity variable given by Cheng and Redner [11, 12]. Finally, solutions for special cases of the polynomial kernel are worked out that provide examples of the general analysis and a comparison with known exact solutions.

2. The fragmentation equation and the Mellin transform

According to Fillipov [1], Ziff and McGrady [8, 9], Peterson [10], Cheng and Redner [11, 12] and Huang *et al* [5], the equation governing the time evolution of the volume distribution of fragmenting particles is given by

$$\frac{\partial n(x,t)}{\partial t} = -\tilde{c}_\alpha x^\alpha n(x,t) + \tilde{c}_\alpha \int_x^\infty y^{\alpha-1} b(x/y) n(y,t)\,\mathrm{d}y \tag{2.1}$$

which is a special case of the master equation of statistical physics as given, for example, by Reichl [22]. In (2.1) we retain the dimensions of the physical quantities where x is the volume of a particle with the dimension of length cubed, t is time and $n(x,t)$ is the uniform spatial concentration of particles per unit particle volume with dimension x^{-2}. The fragmentation equation accounts for the net rate at which the number concentration of particles of volume x increases. In the first term on the right-hand side of (2.1) we assume that the rate at which the concentration of particles of volume x decreases by fragmentation is given by $\tilde{c}_\alpha x^\alpha n(x,t)$, where throughout the analysis we suppose that $\alpha > 0$, which excludes the shattering kind of fragmentation [1, 2, 5, 6]. The second term is the rate at which the concentration of particles of volume x increases due to fragmentation of particles with volumes larger than x and thus the integration over all particle volumes greater than x. The constant $\tilde{c}_\alpha$ has the dimensions of $x^{-\alpha}t^{-1}$ and is usually absorbed in the time variable but we show it explicitly since it has physical significance. The function $b(x/y)$ determines the number of fragments per fragmentation and if $b(x/y)$ is homogeneous of degree zero, as we will assume, then the right-hand side of the fragmentation equation is homogeneous of degree α. This important assumption makes the fragmentation equation invariant under scale transformations.

A recursion equation for the time rate of change of the moments of the particle-volume distribution is obtained by taking moments of (2.1). However, before taking moments we write the equation in another form by substituting the identity

$$b(r) = \frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\int_0^r s b(s)\,\mathrm{d}s \tag{2.2}$$

into (2.1) with the result

$$\frac{\partial n(x,t)}{\partial t} = -\tilde{c}_\alpha (1 - B(1)) x^\alpha n(x,t) + \tilde{c}_\alpha x^{-1} \frac{\partial}{\partial x}\left[x^2 \int_x^\infty y^{\alpha-1} B\left(\frac{x}{y}\right) n(y,t)\,\mathrm{d}y\right] \tag{2.3}$$

where

$$B\left(\frac{x}{y}\right) = B(r) = r^{-2}\int_0^r s b(s)\,\mathrm{d}s \tag{2.4}$$

is dimensionless and homogeneous of degree zero and $B(1) = \int_0^1 r b(r)\,\mathrm{d}r$. Multiplication of (2.3) by x^k and integration over x yields the moment equation

$$\frac{\mathrm{d}M_k(t)}{\mathrm{d}t} = -\tilde{c}_\alpha[(k-1)B_k + 1 - B(1)]M_{k+\alpha}(t) - \tilde{c}_\alpha \lim_{x\to 0} x^{k+1}\int_x^\infty y^{\alpha-1} B\left(\frac{x}{y}\right) n(y,t)\,\mathrm{d}y \tag{2.5}$$

where

$$M_k(t) = \int_0^\infty x^k n(x,t)\,\mathrm{d}x \tag{2.6}$$

are the moments of $n(x, t)$, k is real in the range $k \geqslant 0$ and B_k are the moments of $B(r)$ given by

$$B_k = \int_0^1 r^k B(r)\,\mathrm{d}r \tag{2.7}$$

$$(k-1)B_k = B(1) - \int_0^1 r^k b(r)\,\mathrm{d}r. \tag{2.8}$$

The symbol and the definition for the B-moments are the same as those used by Ziff [13] except the index in (2.7) is shifted by two. The analysis will be restricted to conditions for which the moments of $n(x, t)$ are finite. In particular, the zeroth moment, which is the total number of particles per unit spatial volume, is assumed to be initially finite and remain so for finite times.

For the first moment of $n(x, t)$ we have

$$\frac{\mathrm{d}V(t)}{\mathrm{d}t} = -\tilde{c}_\alpha(1 - B(1))M_{1+\alpha}(t) - \tilde{c}_\alpha \lim_{x\to 0} x^2 \int_x^\infty y^{\alpha-1} B\left(\frac{x}{y}\right) n(y, t)\,\mathrm{d}y \tag{2.9}$$

where the volume fraction V, called volume for short, is the total volume of the fragmenting particles per unit spatial volume. It is dimensionless and has a maximum value of unity. The value $V = 1$ represents the case where the spatial volume is completely filled with the fragmenting material. Since distributions with zero volume will not be considered, we have $0 < V \leqslant 1$. If $B(1) = \int_0^1 rb(r)\,\mathrm{d}r = 1$ and the above boundary term vanishes then the volume of the distribution is constant.

As clarified by Ernst and Szamel [6], if the boundary term in (2.9) does not vanish then, even if the kernel condition $B(1) = 1$ would be satisfied, the volume is not constant. For $\alpha < 0$, Fillipov [1] and McGrady and Ziff [2] have constructed solutions called shattering solutions where $B(1) = 1$ but the boundary term does not vanish so the total volume is not conserved. For example, substituting an exact solution for $\alpha < 0$ given by McGrady and Ziff [2, equation (11)] into the boundary term in (2.9) gives the same rate of change of the volume that they have calculated directly by taking the first moment of the shattering solution.

If the boundary term vanishes for $k \geqslant 0$ and if in addition $B(1) = 1$, as we henceforth assume, then the moments satisfy the differential recursion relation,

$$\frac{\mathrm{d}M_k(t)}{\mathrm{d}t} = -\tilde{c}_\alpha(k-1)B_k M_{k+\alpha}(t) \tag{2.10}$$

which is the Mellin transform of the fragmentation equation with the index shifted by one.

For the zeroth moment (2.10) becomes

$$\frac{\mathrm{d}N(t)}{\mathrm{d}t} = \tilde{c}_\alpha(\tilde{N} - 1)M_\alpha(t) \tag{2.11}$$

where $N(t) = M_0(t)$ is the total particle number density, which for short we call particle number and

$$\tilde{N} = B_0 + 1 = \int_0^1 b(r)\,\mathrm{d}r \tag{2.12}$$

is the number of fragments per fragmentation event. To see that $\tilde{N}$ is given by (2.12) one notes by inspection of (2.1) that $\tilde{c}_\alpha M_\alpha(t)$ is the rate at which particles fragment and therefore the rate at which fragments are created is given by $\tilde{N}\tilde{c}_\alpha M_\alpha(t)$. Then, according to (2.1) with $k = 0$, the difference, given by

$$B_0\tilde{c}_\alpha M_\alpha(t) = \tilde{N}\tilde{c}_\alpha M_\alpha(t) - \tilde{c}_\alpha M_\alpha(t)$$

is the net rate of increase of fragments. It follows that the number of fragments per event, or fragment number as we will call it, is given by (2.12), where we used (2.8). The number of fragments should be finite, so it is necessary that the daughter distribution $b(r)$ be integrable. In order that the particle number increases with time one sees from (2.11) that it is necessary that $B_0 > 0$ (or $\tilde{N} > 1$) and in order to obtain two or more fragments at each fragmentation it is evident from (2.12) that $B_0 \geqslant 1$ (or $\tilde{N} \geqslant 2$). In summary, if the boundary term in (2.9) vanishes, if $B(1) = 1$ and $1 \leqslant B_0 < \infty$, then one may interpret the solution $n(x, t)$ of (2.1) as the evolution in time of a continuous distribution of the volumes of fragmenting particles, where the total particle volume is constant and two or finitely more fragments are formed at each fragmentation.

Taking the characteristic volume to be the initial mean-particle volume, $\nu = V/N_0$ where N_0 is the initial particle number, we define a dimensionless particle volume $\bar{x} = x/\nu$. We take the characteristic time to be $(\tilde{c}_\alpha \nu^\alpha)^{-1}$ where, as is evident from (2.1), $\tilde{c}_\alpha \nu^\alpha$ is the frequency of fragmentation for a particle with mean volume ν. A dimensionless time is defined by $\varsigma = \tilde{c}_\alpha \nu^\alpha t$ and then, in terms of dimensionless variables, the fragmentation equation takes the form

$$\frac{\partial n(\bar{x}, \varsigma)}{\partial \varsigma} = -\bar{x}^\alpha n(\bar{x}, \varsigma) + \int_{\bar{x}}^\infty \bar{y}^{\alpha-1} b\left(\frac{\bar{x}}{\bar{y}}\right) n(\bar{y}, \varsigma)\,\mathrm{d}\bar{y}$$

which is often taken as the starting point of analysis. However, we want to display the physical variables and parameters in the solutions so we will work with (2.1).

3. The similarity solution

The Friedlander ansatz [17–21] gives the desired similarity function form; however, as pointed out by Lushnikov [23], it is not necessary to use the ansatz, rather one may use an invariance argument to derive the function form. Such an argument is given below to show the uniqueness of the Friedlander ansatz and to show the role played by the invariant total-particle volume.

Taking this approach we investigate the transformation of the fragmentation equation under a scale transformation followed by a time translation, namely the transformations

$$t = r^{-\alpha}(t^* + t_0) \qquad x = rx^* \tag{3.1}$$

where x^* and t^* are the new variables, r is a positive real number and t_0 is real. Substitution of the transformation (3.1) into the fragmentation equation (2.1) and relabelling the variables yields the transformed equation.

$$\frac{\partial n_r(x, t)}{\partial t} = -\tilde{c}_\alpha x^\alpha n_r(x, t) + \tilde{c}_\alpha \int_x^\infty y^{\alpha-1} b\left(\frac{x}{y}\right) n_r(y, t)\,\mathrm{d}y$$

where the transformed distribution $n_r(x, t)$ is given by

$$n_r(x, t) = r^2 n[rx, r^{-\alpha}(t + t_0)]. \tag{3.2}$$

We see by inspection that the transformed equation is identical to the original equation.

The factor r^2 makes (3.2) a volume-preserving transformation. To see this we take the first moment of (3.2), which gives

$$V(r) = \int_0^\infty x n_r(x, t)\,\mathrm{d}x = \int_0^\infty y n[y, r^{-\alpha}(t + t_0)]\,\mathrm{d}y = V \tag{3.3}$$

where in the last step the time independence of the first moment was used. Thus, if volume is conserved in the fragmentation process the volume of any solution $n(x, t)$ is invariant under the transformation (3.2).

Since the equation satisfied by $n_r(x, t)$ is exactly the same as the original equation it follows that if $n(x, t)$ is a solution of the fragmentation equation then $n_r(x, t)$ is another, generally different, solution with the same volume. Consequently, the transformation (3.2) provides a way to generate a one-parameter family of solutions from a given solution, where the members of the family all have the same volume, but the other moments may be changed by the transformation. However, we are not interested in generating solutions in this way, rather we seek a special solution, say $n^*(x, t)$, where all the moments are invariant under the group of scale transformations, i.e. where the solution itself is invariant under the scale transformations. Such scale invariant distributions that also satisfy the fragmentation equation for times $t \geqslant 0$ are called similarity solutions or sometimes self-preserving or self-similar solutions [24].

To derive the function form of the similarity solution one notes that if $n^*(x, t)$ is invariant under the scale transformation (3.1) it is independent of r and thus

$$\frac{1}{r}\frac{\mathrm{d}n^*(x, t)}{\mathrm{d}r} = 2n(u, w) + u\frac{\partial n(u, w)}{\partial u} - \alpha w\frac{\partial n(u, w)}{\partial w} = 0 \tag{3.4}$$

where $u = rx$, $w = r^{-\alpha}(t + t_0)$. By the method of characteristics [25] it can be shown that the general solution of (3.4) is

$$n^*(x, t) = A_1 w^{2/\alpha}\phi(A_2 u w^{1/\alpha}) \tag{3.5}$$

where ϕ, referred to here as the reduced distribution, is an arbitrary function and A_1, A_2 and t_0 at this point are undetermined constants. Since r is arbitrary we take $r = 1$ and obtain

$$n^*(x, t) = A_1(t + t_0)^{2/\alpha}\phi[A_2 x(t + t_0)^{1/\alpha}]. \tag{3.6}$$

To determine the constants we take the zeroth and first moments of (3.6) which gives

$$N^* = (A_1/A_2)\mu_0(t + t_0)^{1/\alpha} \qquad V = (A_1/A_2^2)\mu_1 \tag{3.7}$$

where

$$\mu_k = \int_0^\infty z^k\phi(z)\,\mathrm{d}z \tag{3.8}$$

will be called the reduced moments. Eliminating A_1, A_2 and t_0 in (3.6) yields the invariant solution in the form

$$n^*(x, t) = \frac{\mu_1 N^{*2}}{\mu_0^2 V}\phi\left(\frac{\mu_1 N^* x}{\mu_0 V}\right). \tag{3.9}$$

Usually only one normalization condition can be imposed on a solution of a linear homogeneous equation but here it is possible to impose two. It is not necessary but for simplicity we impose the normalization conditions $\mu_0 = 1$ and $\mu_1 = 1$. If both conditions are satisfied then

$$n^*(x, t) = \frac{N^{*2}(t)}{V}\phi(z) \qquad z = \frac{N^*(t)x}{V} \tag{3.10}$$

where the dimensionless similarity variable z is the ratio of the volume of a particle to the instantaneous mean particle volume $V/N^*(t)$. The form of the distribution (3.10) was given as an ansatz by Friedlander and Wang [18] for the homogeneous coagulation equation. The same function form is obtained here for fragmentation because the coagulation equation and the fragmentation equation both have the same translation and scaling invariances. The function $(V/x^2)\Phi(z)$ is also a solution of (3.4); however, since $\Phi(z) = z^2\phi(z)$, this is just a different way of writing the Friedlander form of the solution and is not another independent function. We note that in the above derivation the volume was assumed to be constant but with minor modification the derivation would apply to the volume-changing fragmentation kinetics considered by Fillipov [1], Edwards *et al* [3], Cai *et al* [4] and Huang *et al* [5].

As shown below, substitution of the Friedlander ansatz into the fragmentation equation separates the variables z and t and shows that $\phi(z)$ satisfies an ordinary linear differential–integral equation or, alternatively, shows that $\phi(z)$ satisfies a linear integral equation. The Friedlander ansatz has been used by Fillipov [1] (there called a stationary solution), Peterson [10], Ziff and McGrady [8, 9], Cheng and Redner [11, 12] and Ziff [13]. We call a solution of the fragmentation equation that has the Friedlander form (3.10) a basic similarity solution.

Setting $t_0 = 0$ in (3.6) yields

$$n^*(x,t) \approx A_1 t^{2/\alpha} \phi(A_2 x t^{1/\alpha}) \tag{3.11}$$

which is a limiting form of the basic similarity solution. This form of the distribution has been used in a number of investigations where it is assumed that (3.11) is the continuation to long times of a solution of the initial value problem for the fragmentation equation. However, we will work with the Friedlander form (3.10) so that we may consider the initial value problem and investigate the questions of existence and the continuation of solutions to long times.

4. The time dependence of the moments of the similarity solution

It has been shown by Peterson [10] that the time dependence of the moments of a basic similarity solution can be derived up to numerical values of constants without knowing the reduced distribution $\phi(z)$. The derivation is repeated here in order to introduce some notation for later use and to identify a characteristic time. Substitution of (3.10) into (2.6) gives

$$M_k^*(t) = \mu_k \frac{V^k}{N^{*(k-1)}(t)} \tag{4.1}$$

and with (4.1) and (2.10) we have

$$\frac{\mathrm{d}M_k^*(t)}{\mathrm{d}t} = -\tilde{c}_\alpha (k-1) B_k \mu_{k+\alpha} \frac{V^{(k+\alpha)}}{N^{*(k+\alpha-1)}(t)}. \tag{4.2}$$

For $k = 0$ the integral of (4.2) gives the time dependence of the particle number as

$$\frac{N^*(t)}{N_0} = (1 + \beta^\alpha \varsigma)^{1/\alpha} \tag{4.3}$$

where N_0 is the initial particle number. For conciseness we have used the dimensionless time $\varsigma = \tilde{c}_\alpha \nu^\alpha t$ introduced above and we have identified the important parameter

$$\beta^\alpha = \alpha \mu_\alpha B_0 = \alpha \mu_\alpha (\tilde{N} - 1). \tag{4.4}$$

Now we see the similarity variable is given by

$$z = \frac{N^*(t) x}{V} = \frac{x}{\nu}(1 + \beta^\alpha \varsigma)^{1/\alpha} = \frac{x}{\nu(t)} \tag{4.5}$$

where

$$\nu(t) = \frac{V}{N^*(t)} = \frac{\nu}{(1 + \beta^\alpha \varsigma)^{1/\alpha}}$$

is the instantaneous mean particle volume. With (4.3), the integral of (4.2) yields the time dependence of the other moments as

$$M_k^*(t) = M_k^*(0)(1+\beta^\alpha \varsigma)^{-(k-1)/\alpha} \tag{4.6}$$

where, in accordance with (4.1), $M_k^*(0) = N_0\mu_k \nu^k$ are the initial values of the moments of the similarity solution. If the reduced distribution $\phi(z)$ exists, then (4.6) gives the similarity solution of the moment equation for $t \geqslant 0$ up to the numerical values of $\beta^\alpha \tilde{c}_\alpha \nu^\alpha$ and $M_k^*(0)$. We note that a solution of the form (4.6) for the moments of the similarity solution of the coagulation equation can be derived in the same way.

We see from (4.6) that moments of degree $0 \leqslant k < 1$ increase and the moments of degree $k > 1$ decrease in time. The volume is an invariant of the kinetics and thus $k = 1$ is the separating value for increasing and decreasing moments. This is a general feature of the volume-conserving kinetics, i.e. whatever the fragmentation kernel, if the coefficients B_k are positive the moments less than the first moment increase with time and the moments greater than the first moment decrease with time.

It follows from (4.6) that for long times

$$M_k^*(t) = M_k^*(0)(\beta^\alpha \varsigma)^{-(k-1)/\alpha}\left(1 - \frac{(k-1)}{\alpha}(\beta^\alpha\varsigma)^{-1} + \cdots\right) \tag{4.7}$$

where the series converges if $\beta^\alpha \varsigma > 1$. Thus equation (4.7) gives the continuation of the moments of the similarity solution to long times. From (4.6) and the definition of ς we see that the characteristic time for the kth moment to approach the long-time limit is

$$\tau_k^* = \frac{|k-1|}{\alpha\beta^\alpha \tilde{c}_\alpha \nu^\alpha}. \tag{4.8}$$

With the aid of the definition of $M_k^*(0)$ we have the limit

$$\lim_{t\to\infty} M_k^*(t) = V\mu_k(\beta^\alpha \tilde{c}_\alpha t)^{-(k-1)/\alpha}. \tag{4.9}$$

One sees from (4.8) that for a small initial mean volume it can take a long time for the similarity distribution to reach the limit (4.9). The above results hold for any homogeneous kernel such that the reduced distribution with finite moments exists.

5. The reduced equation and the small and large-z behaviour of the reduced distribution

Up to this point we have proceeded without considering the existence or the construction of the reduced distribution. To reduce the fragmentation equation to an ordinary equation satisfied by $\phi(z)$ we substitute (3.10) into (2.3), use $B(1) = 1$ and use (4.2) with $k = 0$ for $\mathrm{d}N^*/\mathrm{d}t$ and (4.1) for $M_\alpha^*(t)$. Because of the homogeneity in x the explicit time dependence carried by $N^*(t)$ factors out and the resulting equation is

$$B_0\mu_\alpha \frac{\mathrm{d}}{\mathrm{d}z}(z^2\phi(z)) = \frac{\mathrm{d}}{\mathrm{d}z}\left(z^2\int_z^\infty B\left(\frac{z}{w}\right)w^{\alpha-1}\phi(w)\,\mathrm{d}w\right). \tag{5.1}$$

Then by inspection one sees that

$$\phi(z) = \frac{\alpha}{\beta^\alpha}\int_z^\infty B\left(\frac{z}{w}\right)w^{\alpha-1}\phi(w)\,\mathrm{d}w \qquad \alpha > 0 \tag{5.2}$$

is a solution of (5.1), where all the quantities in the equation are dimensionless and the parameter $\beta = (\alpha\mu_\alpha B_0)^{1/\alpha}$ now shows up as a separation constant. We call (5.2) the reduced equation. This integral form of the reduced equation has been studied by Fillipov

[1] and Ziff [13]. We note that β is not arbitrary since, as we will see, it is determined by the kernel parameters.

One can absorb the parameter β and the degree of homogeneity into the similarity variable and work with the scaled similarity variable $\eta = z^\alpha/\beta^\alpha = (\bar{x}^\alpha/\beta^\alpha)(1 + \beta^\alpha \varsigma)$ as the independent variable. Then, carrying out the differentiation in (5.1) and using η as the independent variable, we obtain

$$\eta \frac{\mathrm{d}\bar{\phi}(\eta)}{\mathrm{d}\eta} + \frac{2}{\alpha}\bar{\phi}(\eta) = -\eta\bar{\phi}(\eta) + \frac{1}{\alpha}\int_\eta^\infty b\left(\frac{\eta^{1/\alpha}}{\xi^{1/\alpha}}\right)\bar{\phi}(\xi)\,\mathrm{d}\xi$$

where $\bar{\phi}(\eta) = \phi(z)$, which is the other form of the reduced equation that has been studied. We will work with the integral form of the reduced equation although we do not see an essential advantage over the derivative form. Also, at this point we will use z as the independent variable since we want to see explicitly how β appears in the solutions. Later we will use the scaled similarity variable η to write the solutions of the reduced equation.

5.1. The small-z limit

From the definition of the similarity variable one sees that small z means that the physical particle volume x is small compared to the instantaneous mean volume of the distribution. To obtain the small-z limit we multiply the reduced equation (5.2) by $z^{-\gamma}$, take the limit inside the integral and assume that $\lim_{r\to 0}[r^{-\gamma}B(r)] = \text{constant}$. Then,

$$\begin{aligned}\lim_{z\to 0} z^{-\gamma}\phi(z) &= \frac{\alpha}{\beta^\alpha}\int_0^\infty [\lim_{r\to 0} r^{-\gamma}B(r)]w^{\alpha-1-\gamma}\phi(w)\,\mathrm{d}w \\ &= \frac{\alpha}{\beta^\alpha}[\lim_{r\to 0} r^{-\gamma}B(r)]\mu_{\alpha-1-\gamma}\end{aligned}$$

and thus

$$\lim_{z\to 0}\phi(z) = \text{constant}\,\mu_{\alpha-1-\gamma}z^\gamma. \tag{5.3}$$

This is the small-z limit derived by Cheng and Redner [11, 12]. It follows from (5.3) that it is necessary that $\gamma > -1$ in order to satisfy the normalization condition $\mu_0 = 1$.

5.2. The large-z limit

Large z means that the particle volume x is large compared with the instantaneous mean volume. To investigate the large-z limit we differentiate inside the integral in (5.2) and obtain

$$\frac{\mathrm{d}^2\phi(z)}{\mathrm{d}z^2} = -\frac{\alpha}{\beta^\alpha}\left[z^{\alpha-1}\frac{\mathrm{d}\phi(z)}{\mathrm{d}z} + (\alpha - 1 + B'(1))z^{\alpha-2}\phi(z) + \int_z^\infty B''\left(\frac{z}{w}\right)w^{\alpha-3}\phi(w)\,\mathrm{d}w\right] \tag{5.4}$$

where $B' = \mathrm{d}B/\mathrm{d}r$, $B'' = \mathrm{d}^2B/\mathrm{d}r^2$. The solution for large z has the form

$$\lim_{z\to\infty}\phi(z) = \text{constant}\,z^{B'(1)}\exp\left(-\frac{z^\alpha}{\beta^\alpha}\right)\left\{1 + \mathrm{O}\left[\left(\frac{z}{\beta}\right)^{-\alpha}\right]\right\} \tag{5.5}$$

where, from the definition of $B(r)$, we have $B'(1) = b(1) - 2$. This is the large-z limit derived in another way by Cheng and Redner [11, 12]. In section 6 we will confirm (5.5) by continuing the small-z solution of the reduced equation to large z.

6. Solution of the reduced moment equation and the reduced fragmentation equation

Taking moments (the Mellin transformation) of (5.2) yields

$$\mu_k = \frac{\alpha}{\beta^\alpha}\int_{z=0}^{\infty} z^k \mathrm{d}z \int_{w=z}^{\infty} B\left(\frac{z}{w}\right) w^{\alpha-1}\phi(w)\,\mathrm{d}w$$

where μ_k are the moments defined by (3.8). Changing the way we integrate over w and z we have

$$\mu_k = \frac{\alpha}{\beta^\alpha}\int_{w=0}^{\infty} \mathrm{d}w \int_{z=0}^{w} z^k B\left(\frac{z}{w}\right) w^{\alpha-1}\phi(w)\,\mathrm{d}z.$$

The change of variable $r = z/w$ gives

$$\mu_k = \frac{\alpha}{\beta^\alpha}\int_{w=0}^{\infty} \mathrm{d}w \int_{0}^{1} \mathrm{d}r\, r^k B(r) w^{k+\alpha}\phi(w)$$

and thus

$$\mu_{k+\alpha} = \frac{\beta^\alpha}{\alpha B_k}\mu_k \tag{6.1}$$

where we used (2.7) for B_k. Taking moments of the differential form of the reduced equation also gives the recursion equation (6.1), as it should. We call (6.1) the reduced moment equation. Since the μ-moments should be positive it follows that B_k should also be positive.

Iteration of (6.1) in steps of α yields the solution

$$\mu_{n\alpha} = \frac{\beta^{n\alpha}}{\alpha^n B_0 B_\alpha \cdots B_{(n-1)\alpha}}. \tag{6.2}$$

The solution of the reduced equation (5.2) is given by the inverse Mellin transform of (6.2), i.e. by

$$\phi(z) = \frac{1}{2\pi \mathrm{i}}\int_{Br} z^{-k-1}\mu_k\,\mathrm{d}k \tag{6.3}$$

where Br is the Bromwich path (from $-\mathrm{i}\infty$ to $\mathrm{i}\infty$ to the right of all singularities of μ_k) and μ_k is the solution (6.2) continued to the Bromwich path in the complex k-plane.

6.1. The daughter-fragment distribution

To obtain explicit solutions of the reduced equation we choose a specific form for the daughter-fragment distribution, namely

$$b(r) = r^\gamma(b_0 + b_1 r + \cdots + b_p r^p) \tag{6.4}$$

where $p = 0, 1, \ldots$ is an integer, γ and the coefficients $b_0, b_1, \ldots, b_p$ are real and $0 \leqslant r \leqslant 1$. For brevity, we call $b(r)$ the polynomial daughter distribution and for $p = 0$ we call (6.4) the power-law distribution. With (2.4) and (6.4) we have

$$B(r) = r^\gamma\left(\frac{b_0}{\gamma+2} + \frac{b_1}{\gamma+3}r + \cdots + \frac{b_p}{\gamma+2+p}r^p\right) \tag{6.5}$$

where, as noted above, the constraint $B(1) = 1$ is necessary for the conservation of volume. With (6.5) and (2.7), the moments of $B(r)$ are

$$B_k = \frac{b_0}{(\gamma+2)(k+\gamma+1)} + \frac{b_1}{(\gamma+3)(k+\gamma+2)} + \cdots + \frac{b_p}{(\gamma+2+p)(k+\gamma+1+p)} \tag{6.6}$$

which may be written in the form

$$B_k = \frac{(k+k_1)(k+k_2)\cdots(k+k_p)}{(k+\gamma+1)(k+\gamma+2)\cdots(k+\gamma+1+p)} \tag{6.7}$$

where $-k_1, -k_2, \ldots, -k_p$ are the zeros of B_k. We suppose that the zeros are negative so that $k_1, k_2, \ldots, k_p$ are positive and thus B_k will be positive for $k \geqslant 0$. Within certain constraints, which will be pointed out below, one is free to choose the numerical values of the degree of homogeneity α, the power γ of the leading term of $b(r)$ and the zeros of B_k. Thus, not counting the fragmentation rate constant $\tilde{c}_\alpha$, the theory with a polynomial daughter distribution of degree p is characterized by the numerical values of $p+2$ parameters.

If one regards $\gamma, k_1, k_2, \ldots, k_p$ as independent the coefficients $b_0, b_1, \ldots, b_p$ of the daughter distribution are determined as the solutions of $B_k = 0$, where B_k is given by (6.6). Alternatively if one supposes $b_0, b_1, \ldots, b_p$ are independent then $-k_1, -k_2, \ldots, -k_p$ are solutions of $B_k = 0$. The zeros of B_k are determined as solutions of

$$\frac{b_0}{(\gamma+2)(k+\gamma+1)} + \frac{b_1}{(\gamma+3)(k+\gamma+2)} + \cdots + \frac{b_p}{(\gamma+2+p)(k+\gamma+1+p)} = 0 \tag{6.8}$$

or alternatively, according to (2.8), as solutions of

$$\frac{b_0}{(k+\gamma+1)} + \frac{b_1}{(k+\gamma+2)} + \cdots + \frac{b_p}{(k+\gamma+1+p)} = 1. \tag{6.9}$$

6.2. *The solution μ_k of the reduced moment equation*

For B_k given by (6.7) we have

$$B_0 B_\alpha \cdots B_{(n-1)\alpha} = \frac{(k_1/\alpha)_n (k_2/\alpha)_n \cdots (k_p/\alpha)_n}{\alpha^n ((\gamma+1)/\alpha)_n ((\gamma+2)/\alpha)_n \cdots ((\gamma+1+p)/\alpha)_n} \tag{6.10}$$

where $(\rho)_n = \rho(\rho+1)\cdots(\rho+n-1)$ is the Pochhammer factorial. Using $(\rho)_n = \Gamma(\rho+n)/\Gamma(\rho)$ in (6.10), where Γ is the gamma function, we express the solution of the reduced moment equation in terms of gamma functions as

$$\begin{aligned}\mu_k = \beta^k[&\Gamma((k+\gamma+1)/\alpha)\Gamma((k+\gamma+2)/\alpha)\cdots\Gamma((k+\gamma+1+p)/\alpha)\Gamma(k_1/\alpha)\\ &\times\Gamma(k_2/\alpha)\cdots\Gamma(k_p/\alpha)][\Gamma((\gamma+1)/\alpha)\Gamma((\gamma+2)/\alpha)\cdots\Gamma((\gamma+1+p)/\alpha)\\ &\times\Gamma((k+k_1)/\alpha)\Gamma((k+k_2)/\alpha)\cdots\Gamma((k+k_p)/\alpha)]^{-1}.\end{aligned} \tag{6.11}$$

Taking $k=1$ in (6.11) and recalling that $\mu_1 = 1$ we obtain

$$\beta = \frac{\Gamma((\gamma+1)/\alpha)\Gamma((1+k_1)/\alpha)\Gamma((1+k_2)/\alpha)\cdots\Gamma((1+k_p)/\alpha)}{\Gamma((\gamma+2+p)/\alpha)\Gamma(k_1/\alpha)\Gamma(k_2/\alpha)\cdots\Gamma(k_p/\alpha)} \tag{6.12}$$

and thus β is determined uniquely by the independent fragmentation parameters $\alpha, \gamma, k_1, k_2, \ldots, k_p$.

6.3. *The solution ϕ of the reduced fragmentation equation*

The reduced distribution is obtained by taking the Mellin inverse of the solution (6.11) of the reduced moment equation. Substitution of (6.11) into (6.3) gives

$$\begin{aligned}\bar{\phi}(\eta) = D_p(\alpha,\gamma)&\frac{\alpha}{\beta}\eta^{\gamma/\alpha} G^{p+1,0}_{p,p+1}\\ &\times\left(\frac{k_1-\gamma-1}{\alpha}, \frac{k_2-\gamma-1}{\alpha}, \ldots, \frac{k_p-\gamma-1}{\alpha}; 0, \frac{1}{\alpha}, \frac{2}{\alpha}, \ldots, \frac{p}{\alpha}; \eta\right)\end{aligned} \tag{6.13}$$

where we have introduced the scaled similarity variable $\eta = z^\alpha/\beta^\alpha$, $\bar{\phi}(\eta) = \phi(z)$,

$$D_p(\alpha,\gamma) = \frac{\Gamma(k_1/\alpha)\Gamma(k_2/\alpha)\cdots\Gamma(k_p/\alpha)}{\Gamma((\gamma+1)/\alpha)\Gamma((\gamma+2)/\alpha)\cdots\Gamma((\gamma+1+p)/\alpha)} \tag{6.14}$$

and, according to the definition given by Luke [26],

$$\begin{aligned} &G^{p+1,0}_{p,p+1}\left(\frac{k_1-\gamma-1}{\alpha}, \frac{k_2-\gamma-1}{\alpha}, \ldots, \frac{k_p-\gamma-1}{\alpha}; 0, \frac{1}{\alpha}, \frac{2}{\alpha}, \ldots, \frac{p}{\alpha}; \eta\right) \\ &\qquad = \frac{1}{2\pi\mathrm{i}}\int_{\sigma-\mathrm{i}\infty}^{\sigma+\mathrm{i}\infty} \mathrm{d}k'\eta^{-k'}[\Gamma(k')\Gamma(k'+(1/\alpha))\cdots\Gamma(k'+(p/\alpha))] \\ &\qquad \times[\Gamma(k'+(k_1-\gamma-1)/\alpha)\Gamma(k'+(k_2-\gamma-1)/\alpha)\cdots \\ &\qquad \cdots\Gamma(k'+(k_p-\gamma-1)/\alpha)]^{-1} \end{aligned} \tag{6.15}$$

with $\sigma > 0$ is a Meijer G-function [27]. The G-function is a fundamental transcendent in the theory of generalized hypergeometric functions [26]. In the argument of the G-function it is convenient to use the short notation $(k_p - \gamma - 1)/\alpha$ to stand for $(k_1-\gamma-1)/\alpha, (k_2-\gamma-1)/\alpha, \ldots, (k_p-\gamma-1)/\alpha$ and p/α for $1/\alpha, 2/\alpha, \ldots, p/\alpha$. Thus we will write

$$\begin{aligned} &G^{p+1,0}_{p,p+1}\left(\frac{k_1-\gamma-1}{\alpha}, \frac{k_2-\gamma-1}{\alpha}, \ldots, \frac{k_p-\gamma-1}{\alpha}; 0, \frac{1}{\alpha}, \frac{2}{\alpha}, \ldots, \frac{p}{\alpha}; \eta\right) \\ &\qquad = G^{p+1,0}_{p,p+1}\left(\frac{k_p-\gamma-1}{\alpha}; 0, \frac{p}{\alpha}; \eta\right). \end{aligned}$$

We will show two representations of $G^{p+1,0}_{p,p+1}(\eta)$. The first is a sum of ${}_pF_p$ generalized hypergeometric series and the second is a multiple integral over real variables.

6.4. The G-function as a sum of ${}_pF_p$ generalized hypergeometric series

We can carry out the inversion (6.15) by summing over the residues of the poles of the gamma functions. We first show the details for the special case of $p = 1$, which we call the linear daughter distribution, and then give the result for the polynomial of degree p. For $p = 1$ the definition (6.15) gives

$$G^{2,0}_{1,2}\left(\frac{k_1-\gamma-1}{\alpha}; 0, \frac{1}{\alpha}; \eta\right) = \frac{1}{2\pi\mathrm{i}}\int_{\sigma-\mathrm{i}\infty}^{\sigma+\mathrm{i}\infty} \mathrm{d}k'\,\eta^{-k'}\frac{\Gamma(k')\Gamma(k'+(1/\alpha))}{\Gamma(k'+(k_1-\gamma-1)/\alpha)}. \tag{6.16}$$

If $1/\alpha$ is not an integer there are simple poles of the integrand in (6.16) at $k' = 0, -1, -2, \ldots$; $k' = -1/\alpha, -2/\alpha, -3/\alpha, \ldots$. Summing over the residues of the poles we obtain

$$\begin{aligned} G^{2,0}_{1,2}\left(\frac{k_1-\gamma-1}{\alpha}; 0, \frac{1}{\alpha}; \eta\right) &= \sum_{n=0}^{\infty}\frac{\Gamma(-n+(1/\alpha))}{\Gamma(-n+(k_1-\gamma-1)/\alpha)}\frac{(-1)^n}{n!}\eta^n \\ &\quad + \sum_{n=0}^{\infty}\frac{\Gamma(-n-(1/\alpha))}{\Gamma(-n+(k_1-\gamma-2)/\alpha)}\frac{(-1)^n}{n!}\eta^{1/\alpha+n} \\ &= \frac{\Gamma(1/\alpha)}{\Gamma((k_1-\gamma-1)/\alpha)}\sum_{n=0}^{\infty}\frac{(1-(k_1-\gamma-1)/\alpha)_n}{((\alpha-1)/\alpha)_n}\frac{(-1)^n}{n!}\eta^n \\ &\quad + \frac{\Gamma(-1/\alpha)}{\Gamma((k_1-\gamma-2)/\alpha)}\eta^{1/\alpha}\sum_{n=0}^{\infty}\frac{(1-(k_1-\gamma-2)/\alpha)_n}{((\alpha+1)/\alpha)_n}\frac{(-1)^n}{n!}\eta^n \end{aligned}$$

$$= \frac{\Gamma(1/\alpha)}{\Gamma((k_1-\gamma-1)/\alpha)}\,{}_1F_1\left(1-\frac{k_1-\gamma-1}{\alpha};1-\frac{1}{\alpha};-\eta\right)$$
$$+\frac{\Gamma(-1/\alpha)}{\Gamma((k_1-\gamma-2)/\alpha)}\eta^{1/\alpha}\,{}_1F_1\left(1-\frac{k_1-\gamma-2}{\alpha};1+\frac{1}{\alpha};-\eta\right) \tag{6.17}$$

where

$${}_1F_1(\rho;\sigma;\eta)=\sum_{n=0}^{\infty}\frac{(\rho)_n}{(\sigma)_n}\frac{\eta^n}{n!}$$

is the confluent hypergeometric series. Since ${}_1F_1(\rho;\sigma;-\eta)$ and $\eta^{1-\sigma}\,{}_1F_1(1+\rho-\sigma;2-\sigma;-\eta)$ are solutions of the confluent hypergeometric equation then the above linear combination, $G^{2,0}_{1,2}(\eta)$, is also a solution of the confluent hypergeometric equation.

For the daughter distribution of degree p, if $1/\alpha$ is not an integer, by summing over $p+1$ sequences of residues instead of two, one finds that the G-function is given by

$$G^{p+1,0}_{p,p+1}\left(\frac{k_p-\gamma-1}{\alpha};0,\frac{p}{\alpha};\eta\right)=\sum_{j=0}^{p}\frac{\prod_{i=0,i\neq j}^{p}\Gamma((i/\alpha)-(j/\alpha))\eta^{j/\alpha}}{\prod_{i=1}^{p}\Gamma((k_i-\gamma-1)/\alpha-(j/\alpha))}$$
$$\times{}_pF_p\left(1+\frac{j}{\alpha}-\frac{k_1-\gamma-1}{\alpha},\dots,1+\frac{j}{\alpha}-\frac{k_p-\gamma-1}{\alpha};\right.$$
$$\left.1+\frac{j}{\alpha}-\frac{1}{\alpha},\dots,1+\frac{j}{\alpha}-\frac{j-1}{\alpha},*,1+\frac{j}{\alpha}-\frac{j+1}{\alpha},\dots,1+\frac{j}{\alpha}-\frac{p}{\alpha};-\eta\right) \tag{6.18}$$

where the $*$ notation means that this term is omitted from the product of Pochhammer factorials. The solution $\bar{\phi}(\eta)$ of the reduced equation is given by (6.13) with $G^{p+1,0}_{p,p+1}(\eta)$ given by (6.18).

6.5. *Integral representations of* $G^{p+1,0}_{p,p+1}(\eta)$

In the definition (6.15) of the G-function we express the product of gamma functions in terms of beta functions as

$$[\Gamma(k'+(1/\alpha))\Gamma(k'+(2/\alpha))\cdots\Gamma(k'+(p/\alpha))][\Gamma(k'+(k_1-\gamma-1)/\alpha)$$
$$\times\Gamma(k'+(k_2-\gamma-1)/\alpha)\cdots\Gamma(k'+(k_p-\gamma-1)/\alpha)]^{-1}$$
$$=[\mathrm{B}(k'+(1/\alpha),(k_1-\gamma-2)/\alpha)\mathrm{B}(k'+(2/\alpha),(k_2-\gamma-3)/\alpha)\cdots$$
$$\cdots\mathrm{B}(k'+(p/\alpha),(k_p-\gamma-1-p)/\alpha)][\Gamma((k_1-\gamma-2)/\alpha)$$
$$\times\Gamma((k_2-\gamma-3)/\alpha)\cdots\Gamma((k_p-\gamma-1-p)/\alpha)]^{-1}. \tag{6.19}$$

If $\mathrm{Re}(a)>0, \mathrm{Re}(c)>0$ the beta function is given by

$$\mathrm{B}(a,c)=\frac{\Gamma(a)\Gamma(c)}{\Gamma(a+c)}=\int_0^1 s^{a-1}(1-s)^{c-1}\,\mathrm{d}s. \tag{6.20}$$

Substitution of (6.19) into (6.15) yields

$$G^{p+1,0}_{p,p+1}\left(\frac{k_p-\gamma-1}{\alpha};0,\frac{p}{\alpha};\eta\right)$$
$$=\frac{1}{\Gamma((k_1-\gamma-2)/\alpha)\Gamma((k_2-\gamma-3)/\alpha)\cdots\Gamma((k_p-\gamma-1-p)/\alpha)}$$
$$\times\frac{1}{2\pi\mathrm{i}}\int_{\sigma-\mathrm{i}\infty}^{\sigma+\mathrm{i}\infty}\mathrm{d}k'\,\eta^{-k'}\Gamma(k')\prod_{j=1}^{p}\mathrm{B}\left(k'+\frac{j}{\alpha},\frac{k_j-\gamma-1-j}{\alpha}\right).$$

Using the integral representation of the beta function, changing the order of the beta function integrations and the Bromwich path integration and carrying out the path integration yields

$$
\begin{aligned}
&G^{p+1,0}_{p,p+1}\left(\frac{k_p-\gamma-1}{\alpha};0,\frac{p}{\alpha};\eta\right)\\
&\quad=\frac{1}{\Gamma((k_1-\gamma-2)/\alpha)\Gamma((k_2-\gamma-3)/\alpha)\cdots\Gamma((k_p-\gamma-1-p)/\alpha)}\\
&\quad\times\int_0^1\int_0^1\cdots\int_0^1 \mathrm{d}s_1\,\mathrm{d}s_2\cdots\mathrm{d}s_p\, s_1^{(1/\alpha)-1}(1-s_1)^{((k_1-\gamma-2)/\alpha)-1}\\
&\quad\times s_2^{(2/\alpha)-1}(1-s_2)^{((k_2-\gamma-3)/\alpha)-1}\cdots s_p^{(p/\alpha)-1}(1-s_p)^{((k_p-\gamma-1-p)/\alpha)-1}\\
&\quad\times\exp\left[-\frac{\eta}{(s_1s_2\cdots s_p)}\right].
\end{aligned}
\tag{6.21}
$$

Thus, $\bar{\phi}(\eta)$ is given by (6.13) with $G^{p+1,0}_{p,p+1}((k_p-\gamma-1)/\alpha;0,p/\alpha;\eta)$ given by (6.21).

We suppose that the zeros of B_k are real although, as we will see, it is possible in a natural way to have complex zeros. For real zeros of B_k, if

$$
k_1>\gamma+2 \qquad k_2>\gamma+3 \qquad \cdots \qquad k_p>\gamma+1+p \tag{6.22}
$$

then one sees by inspection of the integrand in (6.21) that the integral over the hypercube exists. Furthermore, with $G^{p+1,0}_{p,p+1}((k_p-\gamma-1)/\alpha;0,p/\alpha;\eta)$ given by (6.21) it is easy to verify that the moments of (6.13) satisfy the reduced moment equation and the normalization conditions $\mu_0=1$, $\mu_1=1$. With a generalization that we will show below, (6.13) contains the similarity solutions that have previously been given for $\alpha>0$.

From (6.7) we have

$$
B_0=\frac{1k_1k_2\cdots k_p}{(\gamma+1)(\gamma+2)\cdots(\gamma+1+p)}. \tag{6.23}
$$

If the conditions on the zeros of B_k given by (6.22) are satisfied the fragment number satisfies the inequality

$$
\tilde{N}=B_0+1>1+\frac{1}{\gamma+1} \tag{6.24}
$$

which shows that for $\gamma\to-1$ a large number of fragments are formed at each fragmentation, independently of the polynomial factor in the daughter distribution.

6.6. An upper bound on ϕ

By inspection we see that by setting $s_1s_2\cdots s_p=1$ in the exponential in (6.21) we obtain an upper bound on the integrand. Then, for $s_1s_2\cdots s_p=1$ the integral is a product of beta functions and with (6.13) we have

$$
\begin{aligned}
\bar{\phi}(\eta)&<D_p(\alpha,\gamma)\frac{\alpha}{\beta}[\mathrm{B}(1/\alpha,(k_1-\gamma-2)/\alpha)\mathrm{B}(2/\alpha,(k_1-\gamma-3)/\alpha)\cdots\\
&\cdots\mathrm{B}(p/\alpha,(k_1-\gamma-1-p)/\alpha)][\Gamma((k_1-\gamma-2)/\alpha)\\
&\times\Gamma_p((k_2-\gamma-3-p)/\alpha)\cdots\Gamma_p((k_p-\gamma-1-p)/\alpha)]^{-1}\eta^{\gamma/\alpha}\exp(-\eta).
\end{aligned}
$$

6.7. Another integral representation

With the change of variables $s_1 = 1/(1+u_1)$, $s_2 = 1/(1+u_2), \ldots, s_p = 1/(1+u_p)$ in (6.21) we obtain

$$\begin{aligned}
G^{p+1,0}_{p,p+1}&\left(\frac{k_p-\gamma-1}{\alpha}; 0, \frac{p}{\alpha}; \eta\right) \\
&= \frac{1}{\Gamma((k_1-\gamma-2)/\alpha)\Gamma((k_2-\gamma-3)/\alpha)\cdots\Gamma((k_p-\gamma-1-p)/\alpha)} \\
&\quad\times \int_0^\infty\int_0^\infty\cdots\int_0^\infty \mathrm{d}u_1\,\mathrm{d}u_2\cdots\mathrm{d}u_p\, u_1^{((k_1-\gamma-2)/\alpha)-1}(1+u_1)^{-(k_1-\gamma-1)/\alpha} \\
&\quad\times u_2^{((k_2-\gamma-3)/\alpha)-1}(1-u_2)^{-(k_2-\gamma-1)/\alpha}\cdots u_p^{((k_p-\gamma-1-p)/\alpha)-1}(1+u_p)^{-(k_p-\gamma-1)/\alpha} \\
&\quad\times \exp[-\eta(1+u_1)(1+u_2)\cdots(1+u_p)]. \qquad (6.25)
\end{aligned}$$

We will use (6.25) to derive the large-η expansion of the reduced distribution.

For the linear daughter distribution (6.25) becomes

$$\begin{aligned}
G^{2,0}_{1,2}\left(\frac{k_1-\gamma-1}{\alpha}; 0, \frac{1}{\alpha}; \eta\right) &= \frac{1}{\Gamma((k_1-\gamma-2)/\alpha)} \\
&\quad\times \int_0^\infty \mathrm{d}u_1\, u_1^{((k_1-\gamma-2)/\alpha)-1}(1+u_1)^{-(k_1-\gamma-1)/\alpha}\exp[-\eta(1+u_1)] \\
G^{2,0}_{1,2}\left(\frac{k_1-\gamma-1}{\alpha}; 0, \frac{1}{\alpha}; \eta\right) &= \exp(-\eta)\psi((k_1-\gamma-2)/\alpha; 1-(1/\alpha); \eta) \qquad (6.26)
\end{aligned}$$

where

$$\begin{aligned}
\psi\left(\frac{k_1-\gamma-2}{\alpha}; 1-\frac{1}{\alpha}; \eta\right) &= \frac{1}{\Gamma((k_1-\gamma-2)/\alpha)} \\
&\quad\times \int_0^\infty \mathrm{d}u_1\, u_1^{((k_1-\gamma-2)/\alpha)-1}(1+u_1)^{-(k_1-\gamma-1)/\alpha}\exp(-\eta u_1) \qquad (6.27)
\end{aligned}$$

sometimes called the ψ function, is one of the fundamental solutions of the confluent hypergeometric equation. A comparison of (6.25) with (6.26) suggests that one may regard $\exp(\eta)G^{p+1,0}_{p,p+1}(\eta)$ as a generalization of the ψ function.

6.8. The small-z limit

As a check on the analysis we confirm that the multiple integral (6.25) satisfies the general small- and large-z limits given in section 5. By expansion of $\exp(-z^\alpha/\beta^\alpha s_1 s_2\cdots s_p)$ in the integrand of (6.25) in a Taylor series at $s_1 = s_2 = \cdots = s_p = 1$ one obtains the small-z limit

$$\begin{aligned}
\lim_{z\to 0}\phi(z) = D_p(\alpha,\gamma)&\frac{\Gamma(1/\alpha)\Gamma(2/\alpha)2\cdots\Gamma(p/\alpha)}{\Gamma((k_1-\gamma-1)/\alpha)\Gamma((k_2-\gamma-1)/\alpha)\cdots\Gamma((k_p-\gamma-1)/\alpha)} \\
&\times\frac{\alpha}{\beta}\left(\frac{z}{\beta}\right)^\gamma \qquad (6.28)
\end{aligned}$$

which is in agreement with Cheng and Redner's limit (5.3).

6.9. The large-z limit

To derive the large-z limit we note that for large z/β the integral (6.15) converges on the Bromwich path for large k. Thus, for large z we may take the path at large real k and use the asymptotic expansion for Γ functions with large argument. Then, to order $1/k$,

$$\frac{\Gamma(k)\Gamma(k+(1/\alpha))\cdots\Gamma(k+(p/\alpha))}{\Gamma(k+((k_1-\gamma-1)/\alpha))\Gamma(k+((k_2-\gamma-1)/\alpha))\cdots\Gamma(k+((k_p-\gamma-1)/\alpha))} = \Gamma\left(k+\frac{\delta}{\alpha}\right)+\mathrm{O}\left(\frac{1}{k}\right) \tag{6.29}$$

where the shift in the argument of the Γ function is given by

$$\delta = -(k_1-\gamma-2)-(k_2-\gamma-3)-\cdots-(k_p-\gamma-1-p). \tag{6.30}$$

By expanding B_k in powers of $1/k$ one obtains

$$B_k = \frac{1}{k} - [\gamma+1-(k_1-\gamma-2)-(k_2-\gamma-3)-\cdots-(k_p-\gamma-1-p)]\frac{1}{k^2}+\mathrm{O}\left(\frac{1}{k^3}\right).$$

Alternatively, expansion of the integrand in (2.7) in a Taylor series at $r=1$ and integration yields the representation

$$B_k = \frac{1}{k} - [B'(1)+1]\frac{1}{k^2}+\mathrm{O}\left(\frac{1}{k^3}\right)$$

where $B(1)=1$ and $B'(1)$ is the derivative of $B(r)$ evaluated at $r=1$. Comparison of terms in the two expansions shows that

$$\delta = B'(1)-\gamma = -(k_1-\gamma-2)-(k_2-\gamma-3)-\cdots-(k_p-\gamma-1-p). \tag{6.31}$$

With (6.15), (6.29)–(6.31) we find that the inverse transform for large z^α/β^α is

$$\phi(z) = D_p(\alpha,\gamma)\frac{\alpha}{\beta}\left(\frac{z}{\beta}\right)^{B'(1)}\exp\left(-\frac{z^\alpha}{\beta^\alpha}\right)\left[1+\mathrm{O}\left(\frac{\beta^\alpha}{z^\alpha}\right)\right] \tag{6.32}$$

which is in agreement with (5.5). Because of the constraints on $k_1, k_2, \ldots, k_p$ we have $B'(1) < \gamma$ so that (6.32) is consistent with the upper bound on ϕ.

7. Examples

We now consider examples for some particular daughter distributions. We start with the simplest.

7.1. The power-law daughter distribution

For $p=0$ in (6.4) we have $b(r)=b_0r^\gamma$. The similarity solution for this case is well known and has been given by Fillipov [1] and Peterson [10]. Nevertheless we consider the solution again in order to comment on the existence, point out its properties and place it in the context of the more general solutions. From (6.5) we see that $B(1)=b_0/(\gamma+2)$ and therefore if $b_0=\gamma+2$ the volume is constant. According to (6.6), $B_0=1/(\gamma+1)$ and thus if $-1<\gamma\leqslant 0$ the fragmentation is binary or larger. Then from (6.13) and (6.15) with $p=0$ we have

$$\bar{\phi}(\eta) = \frac{\alpha}{\Gamma((\gamma+1)/\alpha)\beta}\eta^{\gamma/\alpha}\frac{1}{2\pi\mathrm{i}}\int_{\sigma-\mathrm{i}\infty}^{\sigma+\mathrm{i}\infty}\mathrm{d}k\,\eta^{-k}\Gamma(k).$$

The inverse Mellin transform is

$$\bar{\phi}(\eta) = \frac{1}{\Gamma((\gamma+1)/\alpha)} \frac{\alpha}{\beta} \eta^{\gamma/\alpha} \exp(-\eta) \tag{7.1}$$

where

$$\beta = \frac{\Gamma((\gamma+1)/\alpha)}{\Gamma((\gamma+2)/\alpha)} \qquad \tilde{N} = \frac{\gamma+2}{\gamma+1} \qquad B'(1) = \gamma.$$

This is the form of the solution derived by Peterson [10]. The solution (7.1) is in agreement with the general results for the small- and large-z limits given by (5.3) and (5.5). If $\alpha > 0$ the normalizing conditions $\mu_0 = 1$ and $\mu_1 = 1$ are satisfied and the reduced moments of degree $k \geqslant 0$ are finite and satisfy the reduced moment equation.

With (3.10), (4.3) and (7.1) we have the full similarity solution

$$n^*(x,t) = \frac{N_0^2}{V} \frac{\alpha}{\Gamma((\gamma+1)/\alpha)\beta} \left(\frac{\bar{x}}{\beta}\right)^{\gamma} (1+\beta^{\alpha}\varsigma)^{(\gamma+2)/\alpha} \exp\left[-\frac{\bar{x}^{\alpha}}{\beta^{\alpha}}(1+\beta^{\alpha}\varsigma)\right] \tag{7.2}$$

where $\bar{x} = x/\nu$ is the dimensionless particle volume. We see that the solution depends on the two fragmentation parameters α, γ, the rate constant $\tilde{c}_{\alpha}$ (contained in ς) and the initial mean volume ν.

The solution (7.2) is a one-parameter family of solutions with the initial mean volume ν as parameter. Because of the linearity of the fragmentation equation one can add these solutions to form new solutions. That is essentially how Ziff and McGrady [8] have constructed the general solution of the initial value problem for the power-law daughter distribution.

Expanding (7.2) in powers of $(\beta^{\alpha}\varsigma)^{-1}$ we obtain, in terms of the real time,

$$\begin{aligned} n^*(x,t) = V \frac{\alpha\Gamma((\gamma+2)/\alpha)}{\Gamma^2((\gamma+1)/\alpha)} x^{\gamma} \exp\left(-\frac{x^{\alpha}}{\beta^{\alpha}\nu^{\alpha}}\right) (\tilde{c}_{\alpha}t)^{(\gamma+2)/\alpha} \\ \times \exp(-\tilde{c}_{\alpha}x^{\alpha}t)\left[1 + \left(\frac{\gamma+2}{\alpha\beta^{\alpha}\tilde{c}_{\alpha}\nu^{\alpha}}\right)t^{-1} + \mathrm{O}(t^{-2})\right]. \end{aligned} \tag{7.3}$$

We see that the time constant to approach the long-time limit is $\tau^* = (2+\gamma)/(\alpha\beta^{\alpha}\tilde{c}_{\alpha}\nu^{\alpha})$. Furthermore, as $\gamma \to -1$ the fragment number and β diverge and thus the above time constant and also the time constant for the moments, $\tau_k^* = |k-1|/(\alpha\beta^{\alpha}\tilde{c}_{\alpha}\nu^{\alpha})$, approach zero. As a consequence of the fragment number diverging the particle distribution, $n^*(x,t)$, becomes singular at $x = 0$. On the other hand, it is evident that if $\alpha\tilde{c}_{\alpha}\nu^{\alpha}$ is small enough the characteristic time can be long even if β is large, where we recall that $\tilde{c}_{\alpha}\nu^{\alpha}$ is the initial average fragmentation frequency.

Referring to (7.1) one sees that if $\alpha < 0$ the solution is singular and physically unacceptable. If $\alpha = 0$ the solution of (5.2) is $\phi(z) = \text{constant}\, z^{\gamma-1/B_0}$, which is also singular and thus we do not have a similarity solution for $\alpha \leqslant 0$. A similar argument holds for the general polynomial daughter distribution and we conclude that a physical solution of the reduced equation does not exist for $\alpha \leqslant 0$, as has been noted by other investigators [1, 2, 9, 11].

It is perhaps of interest to compare Peterson's solution with the solutions for $\alpha < 0$, which are not similarity solutions, that have been constructed by Fillipov [1] and McGrady and Ziff [2]. Their solutions, called shattering solutions, show an infinite number of zero-volume (mass) particles created very rapidly with a loss in the total volume of particles. Shattering fragmentation is a cascading process that even for binary fragmentation can very rapidly produce a large number of small particles.

For Peterson's solution, where $-1 < \gamma \leqslant 0$, if $\gamma \to -1$, a large number of smaller particles are created at each fragmentation event without a change of volume, which suggests some sort of explosive process. Even though the fragmentation rate may be slow, if $\gamma \to -1$ the characteristic time for the increase of particle number can be small, in contrast to the cascading shattering-fragmentation process where the characteristic time is small because the rate of fragmentation is large.

7.2. The linear daughter distribution

Next we consider the daughter distribution with the linear polynomial factor,

$$b(r) = r^{\gamma}(b_0 + b_1 r) \tag{7.4}$$

which, as will be discussed below, is a special case of a daughter distribution considered by Ziff and McGrady [9] and Ziff [13]. For $p = 1$ there is one zero in the B-moments at $k = -k_1$ and the b-coefficients are given by

$$b_0 = (2+\gamma)(k_1 - \gamma - 1) \qquad b_1 = -(3+\gamma)(k_1 - \gamma - 2). \tag{7.5}$$

From (6.13) with $p = 1$ and (6.26) we see that the integral form of the solution for the reduced distribution is given by

$$\begin{aligned}
&\bar{\phi}(\eta) = \frac{\Gamma(k_1/\alpha)}{\Gamma((\gamma+1)/\alpha)\Gamma((\gamma+2)/\alpha)} \frac{\alpha}{\beta} \eta^{\gamma/\alpha} \exp(-\eta) \psi\left(\frac{k_1 - \gamma - 2}{\alpha}; 1 - \frac{1}{\alpha}; \eta\right), \\
&\beta = \frac{\Gamma((\gamma+1)/\alpha)\Gamma((k_1+1)/\alpha)}{\Gamma((\gamma+3)/\alpha)\Gamma(k_1/\alpha)} \qquad \tilde{N} = 1 + \frac{k_1}{(\gamma+1)(\gamma+2)} \\
&B'(1) = \gamma - (k_1 - \gamma - 2) \qquad k_1 > \gamma + 2 \qquad k_1 \geqslant (\gamma+1)(\gamma+2)
\end{aligned} \tag{7.6}$$

where $\psi(\eta)$ is given by (6.27). It is easy to check that the normalization of μ_0 and μ_1 are satisfied and that the higher moments satisfy (6.11). If one wants the daughter-fragment number to be a positive integer then one should have $k_1 = B_0(1+\gamma)(2+\gamma)$, where B_0 is a positive integer. For example, the fragmentation is binary if $k_1 = (1+\gamma)(2+\gamma)$, $\gamma > 0$, ternary if $k_1 = 2(1+\gamma)(2+\gamma)$, $2(1+\gamma) > 1$ and so on. The singular behaviour for $\gamma \to -1$, where the fragment number becomes large and τ^* and τ_k^* become small is a general feature and thus also occurs for the linear $b(r)$.

7.2.1. The small-η expansion. As shown, for example, by Luke [26] or Slater [28], if $(\alpha - 1)/\alpha$ is not a negative integer or zero, then the small-η expansion of ψ is

$$\begin{aligned}
\psi\left(\frac{k_1 - \gamma - 1}{\alpha}; 1 - \frac{1}{\alpha}; \eta\right) &= \frac{\Gamma(1/\alpha)}{\Gamma((k_1 - \gamma - 1)/\alpha)}\, {}_1F_1\left(\frac{k_1 - \gamma - 2}{\alpha}; 1 - \frac{1}{\alpha}; -\eta\right) \\
&\quad + \frac{\Gamma(-1/\alpha)}{\Gamma((k_1 - \gamma - 2)/\alpha)} \eta^{1/\alpha}\, {}_1F_1\left(\frac{k_1 - \gamma - 1}{\alpha}; 1 + \frac{1}{\alpha}; -\eta\right).
\end{aligned} \tag{7.7}$$

Then, with the aid of the Kummer transformation, ${}_1F_1(\rho; \sigma; \eta) = \exp(\eta)\, {}_1F_1(\sigma - \rho; \sigma; -\eta)$, we obtain the expansion for $\bar{\phi}(\eta)$ in powers of $(-\eta)$ as

$$\begin{aligned}
\bar{\phi}(\eta) &= \frac{\Gamma(k_1/\alpha)}{\Gamma((\gamma+1)/\alpha)\Gamma((\gamma+2)/\alpha)} \frac{\alpha}{\beta} \eta^{\gamma/\alpha} \\
&\quad \times \left[\frac{\Gamma(1/\alpha)}{\Gamma((k_1 - \gamma - 1)/\alpha)}\, {}_1F_1\left(1 - \frac{k_1 - \gamma - 1}{\alpha}; 1 - \frac{1}{\alpha}; -\eta\right)\right. \\
&\quad \left. + \frac{\Gamma(-1/\alpha)}{\Gamma((k_1 - \gamma - 2)/\alpha)} \eta^{1/\alpha}\, {}_1F_1\left(1 - \frac{k_1 - \gamma - 2}{\alpha}; 1 + \frac{1}{\alpha}; -\eta\right)\right]
\end{aligned} \tag{7.8}$$

which is in agreement with the representation (6.17) of the G-function that was obtained by the residue calculation. The small-η limit of (7.8) is in agreement with the general result given by (6.28). The small-η expansion when $(\alpha-1)/\alpha$ is zero or a negative integer may be found in Luke [26].

7.2.2. The large-η expansion. To obtain the large-η expansion we expand $(1+u)^{-((k_1-\gamma-1)/\alpha)}$ in the integrand of (6.27) in a power series in u. After the change of variable $\xi=\eta u$ we obtain

$$\psi\left(\frac{k_1-\gamma-2}{\alpha};1-\frac{1}{\alpha};\eta\right)=\frac{\eta^{-((k_1-\gamma-2)/\alpha)}}{\Gamma((k_1-\gamma-2)/\alpha)}\int_0^\infty \mathrm{d}\xi\,\xi^{((k_1-\gamma-2)/\alpha)-1}$$
$$\times\exp(-\xi)\left(1-\frac{((k_1-\gamma-1)/\alpha)\xi}{1!}\eta^{-1}\right.$$
$$\left.+\cdots+\frac{(-1)^n((k_1-\gamma-1)/\alpha)_n\xi}{n!}\eta^{-n}+\cdots\right).$$

Integrating term by term gives

$$\psi\left(\frac{k_1-\gamma-2}{\alpha};1-\frac{1}{\alpha};\eta\right)\sim\eta^{-(k_1-\gamma-2)/\alpha}\,{}_2F_0\left(\frac{k_1-\gamma-2}{\alpha},\frac{k_1-\gamma-1}{\alpha};-\eta^{-1}\right)\qquad(7.9)$$

which is the well known analytic continuation of $\psi(\eta)$ to large η. Thus, with (7.6) and (7.9) we obtain the large-η expansion,

$$\bar\phi(\eta)\sim\frac{\Gamma(k_1/\alpha)}{\Gamma((\gamma+1)/\alpha)\Gamma((\gamma+2)/\alpha)}\frac{\alpha}{\beta}\eta^{B'(1)/\alpha}$$
$$\times\exp(-\eta)\,{}_2F_0\left(\frac{k_1-\gamma-2}{\alpha},\frac{k_1-\gamma-1}{\alpha};-\eta^{-1}\right).\qquad(7.10)$$

Since there are two numerator parameters and no denominator parameter the ${}_2F_0$ hypergeometric series diverges for all η; however, it converges asymptotically. The error incurred when the series is truncated at n terms has been derived by Luke [26]. With (3.10), (4.3) and (7.10) the asymptotic expansion of the full similarity solution is

$$n^*(x,t)\sim\frac{N_0^2}{V}\frac{\Gamma(k_1/\alpha)}{\Gamma((\gamma+1)/\alpha)\Gamma((\gamma+2)/\alpha)}\frac{\alpha}{\beta}\left(\frac{\bar x}{\beta}\right)^{B'(1)}(1+\beta^\alpha\varsigma)^{(B'(1)+2)/\alpha}$$
$$\times\exp\left[-\frac{\bar x^\alpha}{\beta^\alpha}(1+\beta^\alpha\varsigma)\right]{}_2F_0\left(\frac{k_1-\gamma-2}{\alpha},\frac{k_1-\gamma-1}{\alpha};-\frac{\beta^\alpha}{\bar x^\alpha(1+\beta^\alpha\varsigma)}\right)\qquad(7.11)$$

where we recall that $\varsigma=\tilde c_\alpha\nu^\alpha t$.

To consider a special case we take $k_1=\gamma+2+\alpha$ in the argument of $\psi((k_1-\gamma-2)/\alpha;1-(1/\alpha);\eta)$. Then, let $k_1=B_0(\gamma+1)(\gamma+2)$ which gives $\tilde N=B_0+1$ fragments and determines γ as a function of B_0. Now the reduced distribution is given by

$$\bar\phi(\eta)=\frac{(\gamma+2)}{\Gamma((\gamma+1)/\alpha)\beta}\eta^{\gamma/\alpha}\exp(-\eta)\psi\left(1;1-\frac{1}{\alpha};\eta\right)$$
$$\beta=\frac{\Gamma((\gamma+1)/\alpha)(\gamma+3)}{\Gamma((\gamma+2)/\alpha)(\gamma+2)}\qquad\gamma=-\frac{3}{2}+\frac{1}{2B_0}+\frac{1}{2}\sqrt{1+\frac{2(1+2\alpha)}{B_0}+\frac{1}{B_0^2}}$$
$$k_1=B_0(\gamma+1)(\gamma+2)\qquad(7.12)$$

where we regard B_0 and α as independent parameters.

For $B_0 = 1$ we have binary fragmentation and $\gamma = -1 + \sqrt{1+\alpha}$ for arbitrary positive α. Then, for example, if $\alpha = \frac{5}{4}$ we have $\gamma = \frac{1}{2}$, $k_1 = \frac{15}{4}$. If $\alpha = 3$ we have $\gamma = 1$, $k_1 = 6$ and (7.12) reduces to a similarity solution for binary fragmentation given by Ziff and McGrady [9].

For $B_0 = 2$ we have ternary fragmentation and $\gamma = -\frac{5}{4} + (\frac{1}{4})\sqrt{9+8\alpha}$. Then, for example, if $\alpha = \frac{7}{8}$, we have $\gamma = -\frac{1}{4}$, $k_1 = \frac{21}{8}$. If $\alpha = 2$, then $\gamma = 0$, $k_1 = 4$. If $\alpha = \frac{27}{8}$, then $\gamma = \frac{1}{4}$, $k_1 = \frac{45}{8}$. The solution (7.12) for $B_0 = 2$, $\alpha = 2$ has been given by Ziff and McGrady [9].

7.3. Ziff's daughter distribution

Ziff's daughter distribution [13] may be written in the form

$$b(r) = r^{\gamma}(b_0 + b_q r^q) \tag{7.13}$$

where q is not necessarily an integer and there is no loss of generality if we suppose that $q > 0$. Regarding k_1 and q as independent parameters, the b-coefficients are determined by

$$\begin{aligned} b_0 &= \frac{1}{q}(2+\gamma)(k_1 - \gamma - 1) \\ b_q &= -\frac{1}{q}(2+q+\gamma)(k_1 - \gamma - 1 - q). \end{aligned} \tag{7.14}$$

When the daughter distribution is given by (7.13), after some obvious changes in (6.11) the Mellin transform solution given in section 6 applies. Now the zero of the B-moments must satisfy $k_1 > \gamma + 1 + q$ and, instead of (7.6), we have

$$\begin{aligned} &\bar{\phi}(\eta) = \frac{\Gamma(k_1/\alpha)}{\Gamma((\gamma+1)/\alpha)\Gamma((\gamma+1+q)/\alpha)}\frac{\alpha}{\beta}\eta^{\gamma/\alpha}\exp(-\eta)\psi\left(\frac{k_1-\gamma-1-q}{\alpha}; 1-\frac{q}{\alpha}; \eta\right) \\ &\beta = \frac{\Gamma((\gamma+1)/\alpha)\Gamma((k_1+1)/\alpha)}{\Gamma((\gamma+2+q)/\alpha)\Gamma(k_1/\alpha)} \qquad \tilde{N} = 1 + \frac{k_1}{(\gamma+1)(\gamma+1+q)} \\ &B'(1) = \gamma - (k_1 - \gamma - 1 - q) \qquad k_1 > \gamma + 1 + q \qquad k_1 \geqslant (\gamma+1)(\gamma+1+q). \end{aligned} \tag{7.15}$$

The small- and large-η expansions and the long-time asymptotic expansion for $n^*(x,t)$ follow in the same way as they did above for the linear case. If $q = 1$ we recover (7.6) and the subsidiary conditions, so it is clear that (7.15) contains the solutions that we showed above for the linear daughter distribution.

To have finite $\tilde{N}$ we must have $\gamma > -1$ and to have two or more fragments per event we must have $k_1 > (\gamma+1)(\gamma+1+q)$. To obtain the solution for an arbitrary integer number of daughter fragments one takes $k_1 = B_0(\gamma+1)(\gamma+1+q)$, $B_0 = 1, 2, 3, \ldots, (\tilde{N}-1)$ in the argument of ψ in (7.15).

The solution (7.15) is simplified if the zero in B_k is at $k_1 = \gamma + 1 + q + \alpha$. Then for arbitrary $\alpha > 0$ and $\gamma > -1$,

$$\bar{\phi}(\eta) = \frac{(\gamma+1+q)}{\Gamma((\gamma+1)/\alpha)\beta}\eta^{\gamma/\alpha}\exp(-\eta)\psi\left(1; 1-\frac{q}{\alpha}; \eta\right) \tag{7.16}$$

where

$$\beta = \frac{\Gamma((\gamma+1)/\alpha)(\gamma+2+q)}{\Gamma((\gamma+1+q)/\alpha)(\gamma+1+q)} \qquad \tilde{N} = 1 + \frac{(\gamma+1+q+\alpha)}{(\gamma+1)(\gamma+2)}$$

which is a similarity solution given by Ziff [13].

As noted by Ziff, the generality introduced by the daughter distribution (7.13) (or (7.4)) removes the restriction $\gamma \leqslant 0$ present in Peterson's solution and large values of γ are thus allowed. The solution for large γ is possibly of interest since a large value of γ has the effect of a smooth, small-r cut-off in the daughter distribution. By inspection of (7.10) and (7.12) one can see the effect that such a cut-off would have on the large- and small-z tails of the similarity distribution. In this connection we note that Cheng and Redner [11, 12] have considered the effect on the particle distribution of a sharp, small-r cut-off in the daughter distribution for a general fragmentation kernel and Huang *et al* [29] have considered a sharp small particle cut-off for a power-law daughter distribution in a linear fragmentation theory that allows mass (or volume) change.

7.4. The quadratic daughter distribution

As a final example we show the reduced distribution for a quadratic polynomial factor in the daughter distribution. In this case we have

$$b(r) = r^{\gamma}(b_0 + b_1 r + b_2 r^2)$$

where there are two zeros of the B-moments. From the conservation of volume (6.5) and equations (6.6) and (6.7) for the B-moments, it follows that

$$\begin{aligned} &b'_0 + b'_1 + b'_2 = 1 \\ &(2\gamma+5)b'_0 + (2\gamma+4)b'_1 + (2\gamma+3)b'_2 = k_1 + k_2 \\ &(\gamma+2)(\gamma+3)b'_0 + (\gamma+1)(\gamma+3)b'_1 + (\gamma+1)(\gamma+2)b'_2 = k_1 k_2 \end{aligned} \tag{7.17}$$

where

$$b'_0 = \frac{b_0}{\gamma+2} \qquad b'_1 = \frac{b_1}{\gamma+3} \qquad b'_2 = \frac{b_1}{\gamma+4}.$$

The solution of (7.17) is given by

$$\begin{aligned} b'_0 &= \tfrac{1}{2}(k_1-\gamma-1)(k_2-\gamma-1) \\ b'_1 &= -(k_1-\gamma-2)(k_2-\gamma-2) \\ b'_2 &= \tfrac{1}{2}(k_1-\gamma-3)(k_2-\gamma-3) \end{aligned} \tag{7.18}$$

where, since the determinant of the matrix of coefficients in (7.17) does not vanish, the solution is unique. One can see that complex conjugate zeros $k_1, k_2 = k_1^*$ will give real values for the b-coefficients. Thus, one could have complex conjugate zeros and still have real and positive B-moments and reduced moments. However, at this point we continue to consider only real zeros.

According to (6.13), for $p = 2$, we have

$$\begin{aligned} \bar{\phi}(\eta) = &\frac{\Gamma(k_1/\alpha)\Gamma(k_2/\alpha)}{\Gamma((\gamma+1)/\alpha)\Gamma((\gamma+2)/\alpha)\Gamma((\gamma+3)/\alpha)}\frac{\alpha}{\beta}\eta^{\gamma/\alpha} \\ &\times G^{3,0}_{2,3}\left(\frac{k_1-\gamma-1}{\alpha}, \frac{k_2-\gamma-1}{\alpha}; 0, \frac{1}{\alpha}, \frac{2}{\alpha}; \eta\right) \\ \beta = &\frac{\Gamma((\gamma+1)/\alpha)\Gamma(k_1/\alpha)\Gamma(k_2/\alpha)}{\Gamma((\gamma+4)/\alpha)\Gamma((1+k_1)/\alpha)\Gamma((1+k_2)/\alpha)} \qquad \tilde{N} = 1 + \frac{k_1 k_2}{(\gamma+1)(\gamma+2)(\gamma+3)} \\ &B'(1) = 3\gamma + 5 - k_1 - k_2 \end{aligned} \tag{7.19}$$

where, according to (6.15),

$$G^{3,0}_{2,3}\left(\frac{k_1-\gamma-1}{\alpha}, \frac{k_2-\gamma-1}{\alpha}; 0, \frac{1}{\alpha}, \frac{2}{\alpha}; \eta\right)$$
$$= \frac{1}{2\pi\mathrm{i}}\int_{\sigma-\mathrm{i}\infty}^{\sigma+\mathrm{i}\infty} \mathrm{d}k\, \eta^{-k} \frac{\Gamma(k)\Gamma(k+(1/\alpha))\Gamma(k+(2/\alpha))}{\Gamma(k+((k_1-\gamma-1)/\alpha))\Gamma(k+((k_2-\gamma-1)/\alpha))} \tag{7.20}$$

with $\sigma > 0$. We take k_1 and k_2 as independent real parameters subject to the constraints that $k_1 > \gamma + 2$, $k_1 \neq \gamma + 3$ and $k_2 > \gamma + 3$, where we have the freedom to take γ large. It is evident that if $k_1 k_2 \geqslant (\gamma+1)(\gamma+2)(\gamma+3)$ the fragmentation is binary or greater.

7.4.1. The small-η expansion. If $1/\alpha$ is not an integer there are simple poles of the Γ-functions at

$$\frac{k+\gamma+1}{\alpha} = -n \qquad \frac{k+\gamma+2}{\alpha} = -n \qquad \frac{k+\gamma+3}{\alpha} = -n \qquad n = 0, 1, 2, \ldots.$$

Summing over the residues in (7.20) we obtain

$$\begin{aligned}\bar{\phi}(\eta) = {}& \frac{\Gamma(k_1/\alpha)\Gamma(k_2/\alpha)}{\Gamma((\gamma+1)/\alpha)\Gamma((\gamma+2)/\alpha)\Gamma((\gamma+3)/\alpha)} \frac{\alpha}{\beta} \eta^{\gamma/\alpha} \\
&\times \Bigg[\frac{\Gamma(1/\alpha)\Gamma(2/\alpha)}{\Gamma((k_1-\gamma-1)/\alpha)\Gamma((k_2-\gamma-1)/\alpha)}\, {}_2F_2 \\
&\times \left(1-\frac{k_1-\gamma-1}{\alpha}, 1-\frac{k_2-\gamma-1}{\alpha}; 1-\frac{1}{\alpha}, 1-\frac{2}{\alpha}; -\eta\right) \\
&+ \frac{\Gamma(-1/\alpha)\Gamma(1/\alpha)}{\Gamma((k_1-\gamma-2)/\alpha)\Gamma((k_2-\gamma-2)/\alpha)} \eta^{1/\alpha}\, {}_2F_2 \\
&\times \left(1-\frac{k_1-\gamma-2}{\alpha}, 1-\frac{k_2-\gamma-2}{\alpha}; 1+\frac{1}{\alpha}, 1-\frac{1}{\alpha}; -\eta\right) \\
&+ \frac{\Gamma(-2/\alpha)\Gamma(-1/\alpha)}{\Gamma((k_1-\gamma-3)/\alpha)\Gamma((k_2-\gamma-3)/\alpha)} \eta^{2/\alpha} \\
&\times {}_2F_2\left(1-\frac{k_1-\gamma-3}{\alpha}, 1-\frac{k_2-\gamma-3}{\alpha}; 1+\frac{2}{\alpha}, 1+\frac{1}{\alpha}; -\eta\right)\Bigg].\end{aligned} \tag{7.21}$$

7.4.2. The large-η expansion. The large-η expansion for $\bar{\phi}$ is obtained in essentially the same way that we did for the expansion of the linear daughter distribution. That is, we start with the integral form of the solution (6.25) for $p = 2$. Then,

$$\begin{aligned}\bar{\phi}(\eta) = {}& \frac{D_2(\alpha,\gamma)}{\Gamma(a_1)\Gamma(a_2)} \frac{\alpha}{\beta} \eta^{\gamma/\alpha} \exp(-\eta) \int_0^\infty \int_0^\infty \mathrm{d}u_1\, \mathrm{d}u_2\, u_1^{a_1-1} \exp(-\eta u_1) u_2^{a_2-1} \\
&\times \exp(-\eta u_2)[(1+u_1)^{c_1-a_1-1}(1+u_2)^{c_2-a_2-1}\exp(-\eta u_1 u_2)]\end{aligned} \tag{7.22}$$

where

$$a_1 = \frac{k_1-\gamma-2}{\alpha} \qquad a_2 = \frac{k_2-\gamma-3}{\alpha} \qquad c_1 = 1-\frac{1}{\alpha} \qquad c_2 = 1-\frac{2}{\alpha}. \tag{7.23}$$

Expanding the function in square brackets in powers of u_1 and u_2, and changing to the variables $\xi_1 = \eta u_1$ and $\xi_2 = \eta u_2$ we obtain, to first order in η^{-1},

$$\begin{aligned}\bar{\phi}(\eta) = {}& \frac{D_2(\alpha,\gamma)}{\Gamma(a_1)\Gamma(a_2)} \frac{\alpha}{\beta} \eta^{(\gamma-a_1-a_2)/\alpha} \exp(-\eta) \int_0^\infty \int_0^\infty \mathrm{d}\xi_1\, \mathrm{d}\xi_2\, \xi_1^{a_1-1} \exp(-\xi_1) \xi_2^{a_2-1} \\
&\times \exp(-\xi_2)\{1 + [(c_1-a_1+1)\xi_1 + (c_2-a_2+1)\xi_2 + \xi_1\xi_2]\eta^{-1} + \cdots\}.\end{aligned}$$

Integrating, factoring out $\Gamma(a_1)\Gamma(a_2)$ and substitution of (7.23) yields

$$\bar{\phi}(\eta j) \sim \frac{\Gamma(k_1/\alpha)\Gamma(k_2/\alpha)}{\Gamma((\gamma+1)/\alpha)\Gamma((\gamma+2)/\alpha)\Gamma((\gamma+3)/\alpha)} \frac{\alpha}{\beta} \eta^{B'(1)/\alpha} \times \exp(-\eta)\left\{1 - \left[\left(\frac{k_1-\gamma-1}{\alpha}\right)\left(\frac{k_1-\gamma-2}{\alpha}\right) + \left(\frac{k_2-\gamma-1}{\alpha}\right)\left(\frac{k_2-\gamma-3}{\alpha}\right) + \left(\frac{k_1-\gamma-2}{\alpha}\right)\left(\frac{k_2-\gamma-3}{\alpha}\right)\right]\eta^{-1} + \cdots\right\} \tag{7.24}$$

where $B' = \gamma - (k_1 - \gamma - 2) - (k_2 - \gamma - 3)$. We see that (7.24) is in agreement with the general limit given by (6.32). In the same way one can derive the large-η expansion to first or higher order in η^{-1} for the solution for the p-degree daughter-fragment distribution.

7.4.3. Complex zeros. As an example of a quadratic daughter distribution where it is possible to have complex conjugate zeros of the B-moments and still have a real daughter distribution, we consider the fragmentation equation in the special form

$$\frac{\partial n(x,t)}{\partial t} = -\int_0^x f(y, x-y)\,\mathrm{d}y\, n(x,t) + 2\int_x^\infty f(x, y-x) n(y,t)\,\mathrm{d}y \tag{7.25}$$

where $f(x, y) = \tilde{C}_\alpha (xy)^{(\alpha-1)/2}$ is the product form of kernel and $\tilde{C}_\alpha$ is a constant with dimensions $(x^\alpha t)^{-1}$. It is easy to show that if a fragmentation kernel is a homogeneous function of x and y and is symmetric in x and y, which are properties of the product kernel, then the solutions of (7.25) conserve the total particle volume and the fragmentation is binary.

Substituting the product kernel into (7.25) we obtain the fragmentation equation in the form (2.1), where

$$\tilde{c}_\alpha = \int_0^1 f(r, 1-r)\,\mathrm{d}r = \tilde{C}_\alpha \mathrm{B}\left(\frac{\alpha+1}{2}, \frac{\alpha+1}{2}\right)$$

and $\mathrm{B}((\alpha+1)/2, (\alpha+1)/2)$ is the beta function and the daughter distribution is

$$b(r) = \frac{2}{\mathrm{B}((\alpha+1)/2, (\alpha+1)/2)} r^{(\alpha-1)/2}(1-r)^{(\alpha-1)/2}. \tag{7.26}$$

If $\alpha = 1, 3, 5, \ldots$, then $b(r)$ is in the polynomial class considered here.

For $\alpha = 1$ we have $b(r) = 2$, which is the $\gamma = 0$, $\tilde{N} = 2$ case of Peterson's solution. For $\alpha = 3$, we have $\mathrm{B}((\alpha+1)/2, (\alpha+1)/2) = 1/3!$ and $b(r) = 12r(1-r)$, which is the linear case considered above with $\gamma = 1$, $k_1 = 6$ where the solution is given by (7.6).

For $\alpha = 5$ we have the quadratic daughter distribution

$$b(r) = 60r^2(1-r)^2$$

where $\gamma = 2$ and $B_k = (k+k_1)(k+k_1^*)/[(k+3)(k+4)(k+5)]$. The zeros are at $k_1 = (13/2) + (\mathrm{i}\sqrt{71}/2)$ and $k_1^* = (13/2) - (\mathrm{i}\sqrt{71}/2)$. We have $\mathrm{Re}(k_1) = \mathrm{Re}(k_2) > 3 + \gamma$ so the condition necessary for the use of the beta functions in the integral form of the solution (6.21) is satisfied. The fragment number is

$$\tilde{N} = 1 + k_1 k_1^*/[\gamma+1)(\gamma+2)(\gamma+3)] = 2$$

as it should be.

As another example where complex zeros arise consider the polynomial distribution $b(r) = 2(2n+1)(1-2r)^{2n}$ [10], which has been proposed in an application to fly ash fragmentation. For $n = 1$ we have $b(r) = 6(1-2r)^2$ which gives $B(1) = 1$ and thus volume is conserved. The B-moments are given by

$$B_k = \frac{(k+k_1)(k+k_1^*)}{(k+1)(k+2)(k+3)}$$

and they are positive. The zeros are at $k_1 = (1/2) + (\mathrm{i}\sqrt{23}/2)$ and $k_1^* = (1/2) - (\mathrm{i}\sqrt{23}/2)$ and one can check that $B_0 = 1$ so the fragmentation is binary. However, $\mathrm{Re}(k_1) = \mathrm{Re}(k_2) < 3+\gamma$ so in this case the condition necessary for the use of the beta functions is not satisfied. This is not to say that there are no solutions of the fragmentation equation for the above daughter distribution only that the solution is not given by the formalism presented here.

7.4.4. Discussion. We have constructed an exact similarity solution of the fragmentation equation for a daughter distribution that is a power times a polynomial function of the fragment volume of degree p, where $p = 0, 1, 2, \ldots$. The solution forms a one-parameter family of similarity solutions with the initial mean volume as the parameter. Although the members of this family could be added to form more general solutions it does not seem that such a superposition would lead to a general solution except for the special case of the power-law distribution. The solution for the monodisperse initial distribution is more fundamental than the similarity solution in the sense that one can construct the general solution by superposition of such solutions. In fact, solutions for the monodisperse initial distribution for the power-law daughter distribution have been constructed by Ziff and McGrady [8] and Huang *et al* [5]. Ziff [13] has also constructed a solution for the monodisperse initial distribution for the volume-conserving daughter distribution

$$b(r) = \frac{(\gamma+2)(\gamma+2+q)}{q} r^\gamma (1-r^q)$$

where $q = \alpha - \gamma - 2$. The construction of this solution was based on a correspondence with the similarity solution for the same daughter distribution. Perhaps the solution for the monodisperse initial distribution for the polynomial daughter distribution can also be constructed by making use of a correspondence with the similarity solution.

For large p it seems that one should be able to approximate a large class of continuous distributions by the polynomial daughter distribution that are consistent with the constraints that we have identified. However, we have not investigated whether this generality would be useful in constructing solutions of physical or mathematical interest.

Redner [30] in a review paper has discussed the comparison of predictions of the linear theory considered here with measurements of particle size distributions and has pointed out some supporting data and also a number of limitations of the theory. We refer to that paper for a broad discussion of the fragmentation problem. However, we do remark, that, as is generally recognized, since the behaviour of the tails of the similarity distributions are especially simple and since for long-times all initial distributions are supposed to tend to the same limiting distribution, measurement of the long-time behaviour of the tails of a particle distribution provides a simple point at which to test the applicability of the theory. Another simple prediction of the theory is the universal behaviour of the moments given here by (4.6)–(4.9). The long-time behaviour of the moments follows only from the assumptions of the linearity of the fragmentation equation and the homogeneity of the fragmentation terms. Thus, measurement of the moments of the distribution would also be a good way to test the basic assumptions of the theory.

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References

[1] Fillipov A F 1961 *Theor. Prob. Appl.* (Engl. Transl.) **4** 275
[2] McGrady E D and Ziff R M 1987 *Phys. Rev. Lett.* **9** 892
[3] Edwards B F, Cai M and Han H A 1990 *Phys. Rev.* A **41** 5755
[4] Cai M, Edwards B F and Han H 1991 *Phys. Rev.* A **43** 656
[5] Huang J, Edwards B F and Levine A D 1991 *J. Phys. A: Math. Gen.* **24** 3967
[6] Ernst M H and Szamel 1993 *J. Phys. A: Math. Gen.* **26** 6085
[7] Said E A and El-Wakil S A 1994 *J. Phys. A: Math. Gen.* **27** 185
[8] Ziff R M and McGrady E D 1985 *J. Phys. A: Math. Gen.* **18** 3027
[9] Ziff R M and McGrady E D 1986 *Macromol. J. Phys.* A **19** 2513
[10] Peterson T W 1986 *Aerosol Sci. Tech.* **5** 93
[11] Cheng Z and Redner S 1988 *Phys. Rev. Lett.* **60** 2450
[12] Cheng Z and Redner S 1990 *J. Phys. A: Math. Gen.* **23** 1923
[13] Ziff R M 1991 *J. Phys. A: Math. Gen.* **24** 2821
[14] Baumann G, Freyberger M, Glockle W G and Nonnenmacher W G 1991 *J. Phys. A: Math. Gen.* **24** 5085
[15] Bak T A and Bak K 1959 *Acta Chem. Scand.* **13** 1997
[16] Smoluchowski M V 1917 *Z. Phys. Chem.* **92** 129
Smoluchowski M V 1917 *Z. Phys. Chem.* **92** 155
[17] Swift D L and Friedlander S K 1964 *J. Colloid Sci.* **19** 621
[18] Friedlander S K and Wang C S 1966 *J. Colloid Interface Sci.* **62** 406
[19] Lai F S, Friedlander S K, Pich J and Hidy G M 1964 *J. Colloid Sci.* **19** 621
[20] Mulholland G W and Baum H R 1977 *J. Colloid Interface Sci.* **62** 406
[21] Friedlander S K 1977 *Smoke Dust and Haze Fundamentals of Aerosol Research* (New York: Wiley)
[22] Reichl L E 1980 *A Modern Course in Statistical Physics* (Austin: University of Texas Press)
[23] Lushnikov A A 1973 *J. Colloid Interface Sci.* **45** 549
[24] Bluman G W and Cole J D 1974 *Similarity Methods for Differential Equations* (New York: Springer)
[25] Ford L R 1955 *Differential Equations* (New York: McGraw-Hill)
[26] Luke L L 1969 *The Special Functions and Their Approximations* vol 1 (New York: Academic)
[27] Meijer C S 1941 *Nederl. Akad. Wetensch. Proc. Ser.* A **44** 81–92
[28] Slater L J 1960 *Confluent Hypergeometric Functions* (Cambridge: Cambridge University Press)
[29] Huang J, Guo X, Edwards B F and Levine A D 1996 *J. Phys. A: Math. Gen.* **29** 7377
[30] Redner S 1990 *Statistical Models for the Fracture of Disordered Media* ch 10, ed H J Hermann and S Roux (Amsterdam: North-Holland)